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Strong Corrections to Inclusive $B \to X\tau\overline{\nu}_\tau$ Decays

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Abstract

We calculate the α_s corrections to the form factors which parameterize the hadronic tensor relevant for inclusive semileptonic $B \to X\tau\overline{\nu}_\tau$ and $\Lambda_b \to X\tau\overline{\nu}_\tau$ decays. We apply our results to the double differential decay rates for the decays of B mesons and polarized Λ_b baryons to polarized τ leptons, presenting them in terms of one-dimensional integrals. Formulas appropriate for insertion into a Monte-Carlo simulation are presented.

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1. Introduction

There has been much recent interest in inclusive semileptonic B decays as a means of measuring the CKM parameters V_{ub} and V_{cb}. These decays are very difficult to calculate due to quark confinement and the breakdown of perturbation theory. However it is possible to improve the behavior of the QCD perturbation series by considering a "smeared" decay rate, in which one integrates in a certain way over the lepton momenta. The infrared singularities responsible for the breakdown of pertubation theory are tamed by integrating appropriately across perturbative thresholds. The idea of smearing to improve the behavior of the QCD pertubation series in the physical region has been used for example to examine the inclusive process $e^+e^- \to$ hadrons [1] For this process one computes averages of the QCD contribution to the vacuum polarization. This smearing method can also be used to describe inclusive semileptonic heavy meson decays $B \to Xl\overline{\nu}$ [2]. In this case there are additional non-perturbative complications, since one must compute the expectation value of quark currents in a B meson rather than the vacuum. This difficulty is controlled by means of a heavy quark expansion. In this paper, we compute perturbative QCD corrections to semi-inclusive semileptonic heavy meson decays involving tau leptons, in the framework of the above smearing method. We will consider the leading term in the heavy quark expansion, so that in practice no non-perturbative matrix elements will be needed.

The semileptonic B meson decays are caused by the Weak Hamiltonian density

$$H_W = -V_{jb}\frac{4G_F}{\sqrt{2}}\overline{q}_j\gamma^\mu P_L b\overline{l}\gamma_\mu P_L\nu, \tag{1.1}$$

where P_L is the chiral projector $\frac{1}{2}(1-\gamma_5)$. The index j denotes either a charm or an up type quark. The inclusive decay rate is given by

$$\frac{d\Gamma}{dq^2dE_\nu dE_\tau} = \frac{|V_{jb}|^2 G_f^2}{2\pi^3}L_{\mu\nu}W^{\mu\nu}. \tag{1.2}$$

$L_{\mu\nu}$ is an easily computed matrix element of leptonic weak currents

$$L_{\mu\nu} = Tr[(\not{p}_l + m_l)\gamma_\mu(\not{p}_\nu)\gamma_\nu P_L], \tag{1.3}$$

and $W_{\mu\nu}$ is the hadronic matrix element

$$W_{\mu\nu} = (2\pi)^3\sum_X \delta^4(p_B - q - p_X)\left\langle B(v,s)\right| J_j^\mu \left|X\right\rangle \left\langle X\right| J_j^\nu \left|B(v,s)\right\rangle, \tag{1.4}$$

with J_j^μ the weak quark current $\overline{u}_j\gamma^\mu P_L b$ and $q = p_l + p_\nu$ the total lepton momentum. Using the optical theorem, $W^{\mu\nu}$ may be expressed as the discontinuity across the cut in the time ordered product of weak currents in a B meson:

$$W^{\mu\nu} = -\frac{1}{\pi} Im T^{\mu\nu}, \tag{1.5}$$

where

$$T^{\mu\nu} = -i \int d^4x \exp(i\mathrm{q}\cdot\mathrm{x}) \langle B| T[J^{\mu^\dagger}(x) J^\nu(0)] |B\rangle . \tag{1.6}$$

The B quark matrix element may be computed using a heavy quark expansion in $\frac{\Lambda_{QCD}}{m_b}$.

In the approach of ref [2] , there are two steps to this expansion. The first consists of an operator product expansion in which one expands about an "on-shell" value for the heavy quark momentum: $p = m_b v + k$ where m_b is the mass parameter of the heavy quark effective lagrangian, v is the four velocity of the B meson, and k is a small residual off-shell momentum. The Wilson coefficients are written as a pertubation series in α_s. The second step is a heavy quark expansion of the expectation values of the local operators in a B meson. The Wilson coefficients are singular at the perturbative threshold, at which the invariant mass of the hadronic decay products is m_j. However, by appropriately smearing the decay rate over lepton energies the operator product expansion can be made well behaved simultaneously with the pertubative QCD expansion. This will be discussed in more detail in section 2. The lowest order term in the $\frac{\Lambda_{QCD}}{m_b}$ expansion is simply the parton model result for the appropriately spin averaged decay of a b quark.

It has been shown that the $\frac{\Lambda_{QCD}}{m_b}$ corrections vanish [2], which lends considerable credence to smeared parton model calculations. The order $(\frac{\Lambda_{QCD}}{m_b})^2$ corrections have been computed at tree level [3], and the resulting parametrization has been applied to the differential decay rate for $\overline{B} \to X\tau\overline{\nu}$ with polarized and unpolarized tau leptons. To order α_s the parton model result for semileptonic decays involving an electron have been calculated [5,6]. For decays involving a massive lepton in the final state, parton model α_s corrections to the total decay rate of $\overline{B} \to X_u\tau\overline{\nu}$ have been computed [7] for vanishing final quark mass and unpolarized tau lepton. Corrections to the differential decay rate $\frac{d\Gamma}{dq^2}$ for B mesons going into polarized or unpolarized tau leptons have also been computed, for massless or massive final quarks, in the $\tau\overline{\nu}$ rest frame [8].

In this paper we compute the matrix element of time ordered weak currents in "off-shell" b quark states to order α_s. We apply this to double differential rates for B meson and Λ_b baryon semi-inclusive semileptonic decays involving either massive or massless final

quarks, and either polarized or unpolarized tau leptons. The off-shell result for the time ordered product is useful should one ever wish to calculate the α_s corrections to the next to leading terms in the $\frac{\Lambda_{QCD}}{m_b}$ expansion. We leave this for future work. By computing the absorbtive part of the on shell value, we arrive at the parton model results for decays involving a tau, in terms of a one dimensional integral over the neutrino energy. This decay rate differs from that involving an electron due to the large τ mass. After a short review of the operator product expansion and the smearing method in section 2, section 3 studies the kinematics of B decays, section 4 gives our result for the B meson triple diferential inclusive decay width, section 5 studies the analytic structure of the form factors, section 6 deals with divergence cancellations, section 7 gives the formulas for polarized Λ_b decays, and section 8 states our conclusions. Our results for B decays are summarized, in a form appropriate for inclusion in a Monte Carlo simulation, in Appendix E.

2. The Smearing Method

In this section we briefly review the procedure of Chay, Georgi, and Grinstein [2] by which one arives at the "smeared" parton model as the leading term in a $\frac{\Lambda_{QCD}}{m_b}$ expansion. The initial step in this procedure is an operator product expansion of the time ordered product of quark weak currents in terms of local operators in the heavy quark effective theory.

$$\int d^4x \exp\{i\mathrm{q}\cdot\mathrm{x}\} T\left(J^\mu(x) J^{\nu^\dagger}(0)\right) = \sum_i C_i(q,v)\hat{O}_{i,v}(0). \tag{2.1}$$

The local operators $\hat{O}_{i,v}(x)$ are made up of light degrees of freedom (gluons and light quarks) and the b heavy quark operator $h_b^v(x)$, related to the full QCD field by

$$h_b^v(x) = \frac{1+\not{v}}{2}\exp(im_b\mathrm{v}\cdot\mathrm{x})b(x), \tag{2.2}$$

with v the four velocity of the B meson. The coefficient functions $C_i(q,v)$ can then be obtained to finite order in α_s by considering the perturbative matrix elements of both sides of equation (2.1) between (unphysical) free quark and gluon states, where the b quark states carry momentum $m_b v + k$. The quark mass used here is the mass parameter of the heavy quark effective lagrangian. All the dependence on the residual momentum k is contained in the matrix elements of local operators containing covariant derivatives of the heavy quark field. The resulting operator product expansion is thus an expansion in off-shell momentum k.

After the expansion has been computed, it is reinserted between B meson states. One might hope to calculate or at least paramaterize the hadronic tensor in this way. However the operator product expansion breaks down near the perturbative threshold. Since the perturbative analytic structure of $T^{\mu\nu}$ is contained in the coefficient functions, the coefficient functions must be singular near thresholds. To see this explicitly, consider the tree level expectation value of the current-current correlator in a b quark:

$$\overline{u}\frac{\gamma^\mu P_L(m_b \not{v} - \not{q} + \not{k} + m_j)\gamma^\nu P_L}{(m_b v - q + k)^2 - m_j^2 + i\epsilon} u. \tag{2.3}$$

The tree level coefficient functions for two quark operators are then obtained by expanding the above expression as a Taylor series in k. The tree level operator product expansion has been considered previously by several authors [3] and the reader is referred to them for details. The important point is that the expansion parameter is

$$\frac{k}{(m_b v - q)^2 - m_j^2 + i\epsilon}. \tag{2.4}$$

The denominator vanishes at threshold, so the operator product expansion breaks down in its vicinity. The discontinuity across the real axis in the complex $\mathrm{v}\cdot\mathrm{q}$ plane consists of an infinite series of derivatives of delta functions which appears to be useless at finite order:

$$T_i = \sum_n c_n \delta^{(n)}\left((m_b v - q)^2 - m_j^2\right), \tag{2.5}$$

where the c_n are matrix elements of the general form

$$c_n = \langle B|\, \overline{h_b^v} D^n h_b^v \,|B\rangle\,. \tag{2.6}$$

The perturbative coefficient functions possess singularities in the wrong places: to arbitrary perturbative order in α_s the threshold singularity is that of the parton model of free quarks and gluons, $(m_b v - q)^2 = m_j^2$, rather than a real hadronic threshold.

These troubles may be avoided by computing $T_i(\mathrm{v}\cdot\mathrm{q}, q^2)$ well away from the physical region. A large imaginary piece to $\mathrm{v}\cdot\mathrm{q}$ acts as an infrared cutoff, allowing one to compute the Wilson coefficients perturbatively [1]. Furthermore, the coefficients get progressively smaller for higher orders in the residual momentum k. Far from the physical region, one expects both the operator product expansion and QCD pertubation theory to be well behaved. Thus one can use them to compute integrals of $W_i(\mathrm{v}\cdot\mathrm{q}, q^2)$ over real $\mathrm{v}\cdot\mathrm{q}$. Due to equation (1.5), such an integral is equivalent to an integral of $T_i(\mathrm{v}\cdot\mathrm{q}, q^2)$ just above the

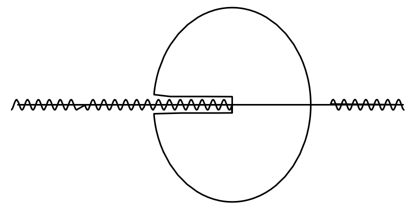

Fig. 1 Analytic structure of $T^{\mu\nu}$ and integration contour in the complex v$\cdot$q plane

cut minus an integral just below. The integration contour can be deformed away from the real axis between the endpoints of the integration. At a distance m_b from threshold in the complex v$\cdot$q plane the effective expansion parameter becomes $\frac{k}{m_b}$. As long as these endpoints are far away from both perturbative and real thresholds, the error due to the integration near the endpoints should be small.

In this paper we compute the differential decay width $\frac{d^3\Gamma}{dE_\tau dq^2 dE_{\overline{\nu}}}$. These three kinematic variables may be varied as long as we treat the final state quark as off-shell. Pertubation theory does not give a sensible result for this rate, but one can obtain sensible results for $\frac{d^2\Gamma}{dE_\tau dq^2}$, provided that E_τ and q^2 are such that the boundaries of the v$\cdot$q integration are not near threshold. Integration over $\mathrm{v}\cdot\mathrm{q} = E_\tau + E_{\overline{\nu}}$ is performed by integrating the triple differential decay width over neutrino energies at fixed tau energy.

In the parton model, the cut in the complex v$\cdot$q plane is given by the B decay threshold

$$\mathrm{v}\cdot\mathrm{q} \leq \frac{m_b^2 + q^2 - m_j^2}{2m_b}. \tag{2.7}$$

There exists another cut in v$\cdot$q, coming from the threshold for weak processes involving two b quarks in the final state. This threshold is given in the parton model by

$$\mathrm{v}\cdot\mathrm{q} \geq \frac{(2m_b + m_j)^2 - m_b^2 - q^2}{2m_b}. \tag{2.8}$$

For the particlar case $m_j = 0$ and $q^2 = m_b^2$, this threshold coincides with the B decay threshold (2.7). In this case, near that point one can not get the operator product expansion to work by integrating past the B decay threshold, because there is no way to close the integration contour in the complex plane far from the region where the perturbative expansion fails. The cuts together with the integration contour are drawn in fig. 1.

We shall compute the B decay rate using only the lowest order term in the operator product expansion. This calculation is equivalent to the parton model calculation of the spin averaged decay rate of a free b quark, where the quark mass is equal to the mass parameter of the heavy quark effective theory. The reason for the equivalence is as follows. The leading term of the operator product expansion is determined from the expectation value of the current correlator in an on shell b quark. This expectation value is of the form $C_j(\mathrm{v}\cdot\mathrm{q}, q^2, \alpha_s)\overline{u}(m_b v)\Gamma_j u(m_b v)$, where Γ_j is some combination of Dirac matrices and u is an on shell spinor. The corresponding term in the operator product expansion is $C_j(\mathrm{v}\cdot\mathrm{q}, q^2, \alpha_s)\overline{h_b^v}\Gamma_j h_b^v$ [2]. This last statement is true to all orders in α_s due to the heavy quark spin symmetry: at leading order in $\frac{\Lambda_{QCD}}{m_b}$ any operator of the form $\overline{h_b^v}\Gamma_j h_b^v$ is a component of a conserved current associated with heavy quark number or heavy quark spin. Furthermore the expectation value of such an operator in a B meson is exactly known by the very same symmetry arguments; the Isgur-Wise function at zero recoil is one. Thus the B meson expectation value averaged over polarizations is equal to the b quark expectation value averaged over spins:

$$T^{\mu\nu} = \frac{1}{2} C_i(\mathrm{v}\cdot\mathrm{q}, q^2) Tr\left(\frac{1+\not{v}}{2}\Gamma_i\right), \tag{2.9}$$

where we have normalized our heavy quark states to v^0.

3. Kinematics

It is convenient to decompose the hadronic tensor $T_{\mu\nu}$ into form factors,

$$\begin{aligned} T^{\mu\nu} = & -g^{\mu\nu}T_1 + v^\mu v^\nu T_2 - i\epsilon^{\mu\nu\alpha\beta}v_\alpha q_\beta T_3 \\ & + q^\mu q^\nu T_4 + (q^\mu v^\nu + q^\nu v^\mu)T_5 \\ & + (q^\mu v^\nu - q^\nu v^\mu)T_6. \end{aligned} \tag{3.1}$$

T_6 vanishes due to the time reversal invariance of the strong interactions: $T^{\mu\nu}(v,q) = T_{\nu\mu}(v'q')$ where $v'^\mu = v_\mu$ and $q'^\mu = q_\mu$. $W_{\mu\nu}$ may be similarly decomposed. The decay rate for spin averaged B's is given by

$$\begin{aligned} \frac{d^3\Gamma}{dq^2 dE_\tau dE_\nu} = \frac{|V_{jb}|^2\, G_F^2}{2\pi^3} \Big[& W_1\left(q^2 - m_\tau^2\right) + W_2\left(2E_\nu E_\tau - \frac{q^2 - m_\tau^2}{2}\right) \\ & + W_3\left((E_\tau - E_\nu)(q^2 - m_\tau^2) - 2E_\nu m_\tau^2\right) \\ & + m_\tau^2\left(W_4(\frac{q^2 - m_\tau^2}{2}) + W_5(2E_\nu)\right) \Big]. \end{aligned} \tag{3.2}$$

[2] Operators which contain gluons or light quark pairs are sub-leading. Thus, weak annihilation of the b quark and light antiquark is suppressed in the $m_b \to \infty$ limit.

For $\pm$ helicity tau's we have:

$$\frac{d^3\Gamma_\pm}{dq^2 dE_\tau dE_\nu} = \frac{1}{2}\frac{d^3\Gamma}{dq^2 dE_\tau dE_\nu}\left(1 \pm \frac{E_\tau}{\sqrt{E_\tau^2 - m_\tau^2}}\right)$$
$$\pm \frac{|V_{jb}|^2\, G_F^2}{4\pi^3} m_\tau^2 \frac{E_\tau}{\sqrt{E_\tau^2 - m_\tau^2}}\Big[-2W_1 - W_2 - 2W_3\left(\sqrt{E_\tau^2 - m_\tau^2}\cos\alpha + E_\nu\right)$$
$$+ W_4\left(m_\tau^2 + 2E_\tau(\sqrt{E_\tau^2 - m_\tau^2}\cos\alpha - E_\tau)\right)$$
$$+ W_5\left(2(\sqrt{E_\tau^2 - m_\tau^2}\cos\alpha - E_\tau)\right)\Big], \tag{3.3}$$

where the angle between the tau and neutrino trajectories is given by

$$\cos\alpha = \frac{m_\tau^2 - q^2 + 2E_\nu E_\tau}{2E_\nu\sqrt{E_\tau^2 - m_\tau^2}}. \tag{3.4}$$

Note that this expression reduces to eq. (3.2) when summed over tau helicities. The leptonic variables E_τ, q^2 and $E_\nu = \mathrm{v}\cdot\mathrm{q} - E_\tau$ are constrained to the region given by

$$m_\tau \leq E_\tau \leq \frac{m_B}{2}(1 - \frac{m_X^2 - m_\tau^2}{m_B^2}), \tag{3.5}$$

$$m_\tau^2 \leq \mathrm{q}^2 \leq \frac{(E_\tau + \sqrt{E_\tau^2 - m_\tau^2})(m_B^2 - m_X^2 - 2E_\tau m_B) + m_\tau^2 m_B}{m_B - E_\tau - \sqrt{E_\tau^2 - m_\tau^2}}, \tag{3.6}$$

$$E_\tau + \frac{q^2 - m_\tau^2}{2(E_\tau + \sqrt{E_\tau^2 - m_\tau^2})} \leq \mathrm{v}\cdot\mathrm{q} \leq \mathrm{Min}[E_\tau + \frac{q^2 - m_\tau^2}{2(E_\tau - \sqrt{E_\tau^2 - m_\tau^2})}, \frac{m_B^2 + q^2 - m_X^2}{2m_B}]. \tag{3.7}$$

m_B is the B meson mass, and m_X the mass of the lightest hadron containing the final state u or c quark . With the usual assumption $m_B > m_b$, the allowed region of integration for $\mathrm{v}\cdot\mathrm{q}$ includes the parton model theshold (2.7). The upper limit for $\mathrm{v}\cdot\mathrm{q}$ shows a new feature of tau kinematics; for a massless final lepton the upper limit is simply the threshold value (2.7). The lower and upper bounds in (3.7) correspond to parallel and anti-parallel τ and ν_τ trajectories.

In the case of a decaying polarized Λ_b we can parameterize the spin-dependent part of $T^{\mu\nu}$ in terms of nine additional form factors,

$$T_S^{\mu\nu} = -\mathrm{s}\cdot\mathrm{q}\Big[-g^{\mu\nu}G_1 + v^\mu v^\nu G_2 - i\epsilon^{\mu\nu\alpha\beta}v_\alpha q_\beta G_3 + q^\mu q^\nu G_4$$
$$+ (q^\mu v^\nu + q^\nu v^\mu)\, G_5\Big] + (s^\mu v^\nu + s^\nu v^\mu)\, G_6 + (s^\mu q^\nu + s^\nu q^\mu)\, G_7 \tag{3.8}$$
$$+ i\epsilon^{\mu\nu\alpha\beta}v_\alpha s_\beta G_8 + i\epsilon^{\mu\nu\alpha\beta}q_\alpha s_\beta G_9,$$

where s is the spin vector of the Λ_b baryon in the Λ_b rest frame. Note that there are no terms of the form $i(q^\mu\epsilon^{\nu\alpha\beta\lambda}v_\alpha q_\beta s_\lambda-(\mu\to\nu))$ or $i(v^\mu\epsilon^{\nu\alpha\beta\lambda}v_\alpha q_\beta s_\lambda-(\mu\to\nu))$. They can be shown to vanish using the identity

$$g^{\mu\nu}\epsilon^{\alpha\beta\lambda\sigma}-g^{\mu\alpha}\epsilon^{\nu\beta\lambda\sigma}+g^{\mu\beta}\epsilon^{\alpha\nu\lambda\sigma}-g^{\mu\lambda}\epsilon^{\nu\alpha\beta\sigma}+g^{\mu\sigma}\epsilon^{\nu\alpha\beta\lambda}=0.$$

The differential decay rate in this case has additional terms that depend on the spin of the Λ_b:

$$\begin{aligned}
\frac{d^4\Gamma_\pm}{dq^2dE_\tau dE_\nu d\cos\theta} &= \frac{1}{2}\,\frac{d^3\Gamma_\pm}{dq^2dE_\tau dE_\nu}\\
&+\cos\theta\Bigg[\frac{1}{2}\,\frac{d^3\Gamma_\pm^{(W_i\to G_i)}}{dq^2dE_\tau dE_\nu}\quad(E_\nu\cos\alpha+\sqrt{E_\tau^2-m_\tau^2})\\
&\qquad+\frac{|V_{jb}|^2\,G_F^2}{8\pi^3}\Bigg[\left(1\pm\frac{E_\tau}{\sqrt{E_\tau^2-m_\tau^2}}\right)\Big(\,G_6\left(-2E_\nu(E_\tau\cos\alpha)+\sqrt{E_\tau^2-m_\tau^2}\right)\\
&\qquad\qquad-G_7\left(2m_\tau^2E_\nu\cos\alpha\right)-G_8\left(2E_\nu(E_\tau\cos\alpha-\sqrt{E_\tau^2-m_\tau^2})\right)\\
&\qquad\qquad-G_9\left((q^2-m_\tau^2)(E_\nu\cos\alpha-\sqrt{E_\tau^2-m_\tau^2})+2m_\tau^2E_\nu\cos\alpha\right)\Big)\\
&\qquad\qquad\pm m_\tau^2\frac{2E_\nu}{\sqrt{E_\tau^2-m_\tau^2}}\Big(G_6\left(\cos\alpha\right)+G_7\left(E_\tau\cos\alpha-\sqrt{E_\tau^2-m_\tau^2}\right)\\
&\qquad\qquad\qquad+G_8\left(\cos\alpha\right)+G_9\left(\cos\alpha(E_\nu+E_\tau)\right)\Big)\Bigg]\Bigg],
\end{aligned}\tag{3.9}$$

where θ is the angle between the spin vector of the Λ_b and the τ three-momentum, and where in the second term $d^3\Gamma_\pm^{(W_i\to G_i)}$ is identical to $d^3\Gamma_\pm$, except that all the W_i's are replaced by the spin-dependent G_i's.

Note that when we integrate the polarized cross section (3.9) over $\cos\theta$ the term proportional to $\cos\theta$ vanishes, while the constant term gets multiplied by a factor of 2 and so reproduces eq. (3.3).

4. B meson decay

To compute the time ordered product of eq. (1.5), we must evaluate the graphs in figs. 2,3,4 and 5. We do this by first separating the Dirac structure from the loop integrals, evaluating the loop integrals in terms of a function we call S, and reducing the Dirac algebra with the help of the heavy quark effective theory. We present analytic formulas for S and express the form factors T_i in terms of S and its derivatives. After that, it is a matter

of numerical integration to generate plots at arbitrary values of q^2 and E_τ, with kinematic cuts, or with additional refinements. Such refinements might include corrections due to the W propagator [6], or m_j/m_b corrections arising from expressing the width in terms of quark masses evaluated at one scale instead of on-shell [9]. In addition, higher order corrections due to these graphs (such as the α_s correction to the $\frac{1}{m_b^2}$ operator $\langle B| \, (\mathrm{i}D)^2 \, |B\rangle$) may be computed by substituting $p \to m_b v + k$ and expanding in $\frac{k}{m\,b}$. Strong corrections to decays of polarized $\Lambda_b' s$ may also be found straightforwardly by evaluating the Dirac traces between spinors instead of averaging over spin.

We do our calculation in minimal subtraction and Feynman gauge, with a gluon mass λ to regulate infrared divergences. For a massless final state, there is considerable simplification if one uses dimensional regularization instead of a gluon mass [10], but for $B \to cl\overline{\nu}$, the simplification is less obvious. For the charm case, the amplitudes may be expressed in terms of generalized hypergeometic functions, which can be numerically integrated over the charm momentum and analytically integrated over the lepton momenta [11]. However, if lepton spectra are desired, this trick for analytic integration over lepton momenta is not useful.

We devide our calculation into several parts. The total form factor is expressed as the sum of the contributions of various diagrams

$$T_i = T_i^0 + T_i^{self} + T_i^{vert} + T_i^{box}.$$

4.1. Heavy Quark Self Energy corrections

The quark matrix element of the tree level graph of fig. 2 is

$$\overline{u}_b(p) \frac{\gamma^\mu (\not{p} - \not{q}) \gamma^\nu P_L}{[(p-q)^2 - m_j^2 + \mathrm{i}\epsilon]} u_b(p), \tag{4.1}$$

where p is the momentum of the external b quark, q is the W momentum, and $u_b(p)$ is the b quark spinor. For the parton model result, we will eventually take $p = m_b v$, with $v^2 = 1$.

Wave-function renormalization of the external heavy quark lines modify the tree level diagram by the multiplicative factor

$$T^{\mu\nu}_{tree} \to T^{\mu\nu}_{tree}(1 + \delta Z_b),$$

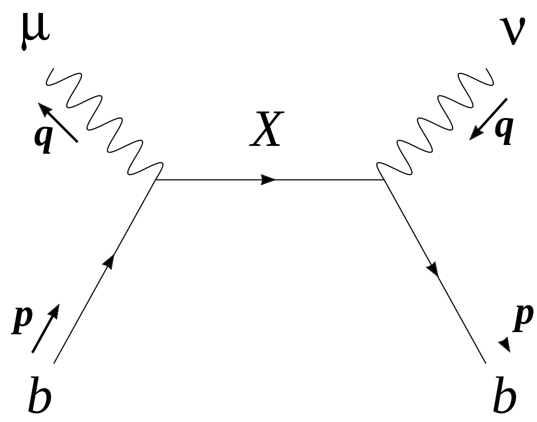

Fig. 2 Tree level contribution to inclusive semi-leptonic B decay.

where $\delta Z_b = \frac{\partial}{\partial \not{p}}\Sigma|_{\not{p}=m_b}$, and $-\mathrm{i}\Sigma$ is the self-energy loop of a b quark with momentum p. This loop gives, for $D = 4-\epsilon$ dimensions,

$$-\mathrm{i}\Sigma = -\frac{\mathrm{i}\alpha_s}{3\pi}[(\epsilon-2)\not{p}B_{00}(p,0,m_b) + (4-\epsilon)m_b B_0(p,0,m_b)],$$

where

$$\begin{aligned} B_0(p,0,m_b) &= \frac{-\mathrm{i}}{\pi^2}\int \frac{2\pi^{D-4}d^D l}{[(l-p)^2-\lambda^2+\mathrm{i}\epsilon][l^2-m_b^2+\mathrm{i}\epsilon]} \\ &= \frac{2}{\epsilon}+\ln(4\pi)-\gamma+2+\mathrm{i}\pi\frac{p^2-m_b^2}{p^2} \\ &\quad -\frac{m_b^2}{p^2}\ln(\frac{m_b^2}{\mu^2})-\frac{p^2-m_b^2}{p^2}\ln(\frac{p^2-m_b^2}{\mu^2}) \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} p^\alpha B_{00}(p,0,m_b) &= \frac{-\mathrm{i}}{\pi^2}\int \frac{2\pi^{D-4}d^D l \quad l^\alpha}{[(l-p)^2-\lambda^2+\mathrm{i}\epsilon][l^2-m_b^2+\mathrm{i}\epsilon]} \\ &= p^\alpha[-\frac{1}{\epsilon}-\frac{1}{2}\ln 4\pi+\frac{1}{2}\gamma-1+\mathrm{i}\pi\frac{m_b^4-p^4}{2p^4} \\ &\quad +\frac{m_b^4}{2p^4}\ln\frac{m_b^2}{\mu^2}-\frac{m_b^2}{2p^2}+\frac{p^4-m_b^4}{2p^4}\ln\frac{p^2-m_b^2}{\mu^2}]. \end{aligned} \tag{4.3}$$

We may set the gluon mass to zero for the self-energy, which is infrared finite, but for the derivative of the self-energy the gluon mass λ is needed to regulate the infrared divergence. In this case, we get

$$\delta Z_b = \frac{\partial}{\partial \not{p}}\Sigma|_{\not{p}=m_b} = \frac{\alpha}{3\pi}\left[-\frac{2}{\epsilon}-\ln 4\pi+\gamma-4+\ln\frac{m_b^2}{\mu^2}-2\ln\frac{\lambda^2}{m_b^2}\right].$$

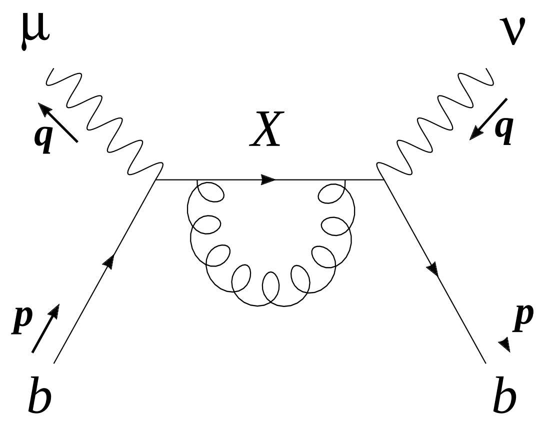

Fig. 3 Final quark self-energy diagram.

The resulting form factors are

$$
\begin{aligned}
T_1^0 &= \frac{1}{2\Delta_0}(m_b - \mathrm{v}\cdot\mathrm{q})(1+\delta Z_b)\\
T_2^0 &= \frac{1}{\Delta_0}m_b((1+\delta Z_b)\\
T_3^0 &= \frac{1}{2\Delta_0}((1+\delta Z_b)\\
T_4^0 &= 0\\
T_5^0 &= -\frac{1}{2\Delta_0}((1+\delta Z_b),
\end{aligned}
\tag{4.4}
$$

where $\Delta_0 = (p-q)^2 - m_j^2 + \mathrm{i}\epsilon$.

4.2. Final Quark Self Energy

Since we are using the optical theorem, the light c or u quark self-energy is not incorporated into wave-function renormalization, but instead arises as a separate graph. These contributions to the form factors enter in the same proportions as the tree result, since self-energy corrections don't alter the Lorentz structure of the leading order operator. The self-energy graph of fig. 3, which also accounts for final state radiation, gives

$$
\frac{\mathrm{i}\alpha_s}{3\pi}\frac{\overline{u}_b(p)\gamma^\mu(\not{p}-\not{q})\gamma^\nu P_L u_b(p)}{(\Delta_0)^2}\big[2m_j^2(4-\epsilon)B_0(p-q,0,m_j)+
((p-q)^2+m_j^2)(\epsilon-2)B_{00}(p-q,0,m_j)\big].
$$

It gives a form factor contribution of

$$
\begin{aligned}
T_1^{self} &= \frac{\alpha_s}{3\pi}\,\frac{1}{2\Delta_0^2}\,(m_b - \mathrm{v}\cdot\mathrm{q}\,)\,\mathcal{A}[(p-q)^2, m_j^2] \\
T_2^{self} &= \frac{\alpha_s}{3\pi}\,\frac{1}{\Delta_0^2}\,m_b\,\mathcal{A}[(p-q)^2, m_j^2] \\
T_3^{self} &= \frac{\alpha_s}{3\pi}\,\frac{1}{2\Delta_0^2}\,\mathcal{A}[(p-q)^2, m_j^2] \\
T_4^{self} &= 0 \\
T_5^{self} &= \frac{\alpha_s}{3\pi}\,\frac{-1}{2\Delta_0^2}\,\mathcal{A}[(p-q)^2, m_j^2],
\end{aligned}
\tag{4.5}
$$

where

$$
\begin{aligned}
\mathcal{A}[(p-q)^2, m_j^2] = &\left(\frac{2}{\epsilon} - \gamma + \ln(4\pi\mu^2)\right)(-(p-q)^2 + 7m_j^2) - (p-q)^2 + 10m_j^2 - \frac{m_j^4}{(p-q)^2} \\
&- \left(\frac{m_j^6}{(p-q)^4} - 7\frac{m_j^4}{(p-q)^2} - (p-q)^2 + 7m_j^2\right)\ln((m_j+\lambda)^2 - (p-q)^2 - i\epsilon) \\
&+ \left(\frac{m_j^6}{(p-q)^4} - 7\frac{m_j^4}{(p-q)^2}\right)\ln(m_j^2).
\end{aligned}
\tag{4.6}
$$

The optical theorem tells us that to obtain the B decay rate we need to compute the imaginary part of the T_i's. In the case of the heavy quark self-energy, as δZ_b is real, there is only one contribution, coming from

$$
Im\{\frac{1}{(x+i\epsilon)}\} = -\pi\delta(x). \tag{4.7}
$$

On the other hand, $Im(T^{self})$ has two contributions, a first one from

$$
Im\{\frac{1}{(x+i\epsilon)^2}\} = \pi\delta'(x), \tag{4.8}
$$

and a second one from the imaginary part of the function $\mathcal{A}[(p-q)^2, m_j^2]$. Those two contributions correspond to two different cuts in the Feynman diagram. In the first, we cut the final quark propagator near the weak vertex and put it on-shell with the Dirac delta. The virtual gluon renormalizes the light quark wave-function. It is easy to see, by integrating the δ and δ' functions over v·q, that the resulting initial and final quark wave-function renormalizations are identical, after the obvious interchange $m_b \leftrightarrow m_j$. The

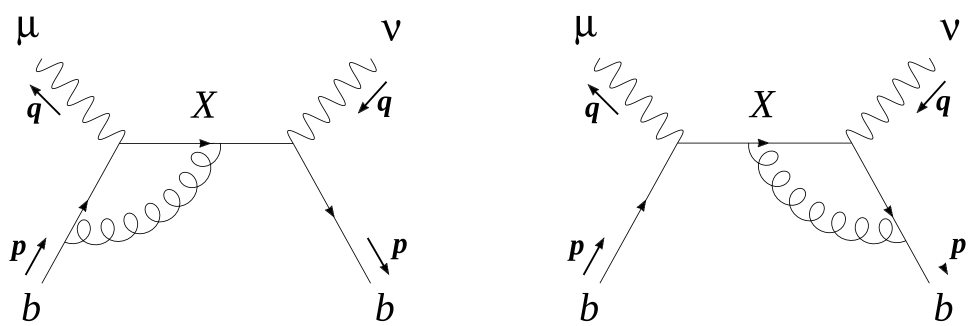

Fig. 4 Vertex correction diagrams.

second contribution corresponds to cutting through the loop. In this case, the gluon is real, corresponding to final state bremsstrahlung. The fact that $\mathcal{A}$ only has a non-zero imaginary part when $(p-q)^2 \geq (m_j+\lambda)^2$ immediately implies the parton model branch cut (2.7). $\mathcal{A}$ has an additional dependence on the gluon mass which cancels the infrared divergences that arise from the vertex corrections. The analogous infrared divergence from bremsstrahlung off the initial quark is contained in the box diagram, fig. 5.

4.3. Vertex Corrections

The virtual gluon correction to the weak vertex is

$$-\frac{\mathrm{i}4g^2}{3}\int\frac{\mathrm{d}^D l}{(2\pi)^D}\frac{\gamma^\alpha(\not{l}+m_b)\gamma^\mu P_L(\not{l}-\not{q}+m_j)\gamma_\alpha}{[(l-p)^2-\lambda^2+\mathrm{i}\epsilon][(l-q)^2-m_j^2+\mathrm{i}\epsilon][l^2-m_b^2+\mathrm{i}\epsilon]}. \tag{4.9}$$

This enters into both vertex graphs in fig. 4, giving

$$\frac{\mathrm{i}\alpha_s}{3\pi\Delta_0}\overline{u}_b(p)[N^{(V)}C+N_\alpha^{(V)}C^\alpha+N_{\alpha\beta}^{(V)}C^{\alpha\beta}]u_b(p), \tag{4.10}$$

where

$$C^{[\cdot,\alpha,\alpha\beta]}=\frac{-\mathrm{i}}{\pi^2}\int\frac{2\pi^{D-4}d^Dl\quad[1,l^\alpha,l^\alpha l^\beta]}{[l^2-m_b^2+\mathrm{i}\epsilon][(l-q)^2-m_j^2+\mathrm{i}\epsilon][(l-p)^2-\lambda^2+\mathrm{i}\epsilon]} \tag{4.11}$$

and

$$\begin{aligned}
N^{(V)} &= m_b\gamma^\lambda\gamma^\mu(\gamma_\lambda m_j^2-\not{q}\gamma_\lambda\not{p}')\gamma^\nu P_L+m_b\gamma^\mu(\gamma^\lambda m_j^2-\not{p}'\gamma^\lambda\not{q})\gamma^\nu P_L\gamma_\lambda\\
N_\alpha^{(V)} &= \gamma^\lambda\gamma_\alpha\gamma^\mu(\gamma_\lambda m_j^2-\not{q}\gamma_\lambda\not{p}')\gamma^\nu P_L+m_b\gamma^\lambda\gamma^\mu\gamma_\alpha\gamma_\lambda\not{p}'\gamma^\nu P_L\\
&\quad+\gamma^\mu(\gamma^\lambda m_j^2-\not{p}'\gamma^\lambda\not{q})\gamma^\nu P_L\gamma_\alpha\gamma_\lambda+m_b\gamma^\mu\not{p}'\gamma^\lambda\gamma_\alpha\gamma^\nu P_L\gamma_\lambda\\
N_{\alpha\beta}^{(V)} &= \gamma^\lambda\gamma_\alpha\gamma^\mu\gamma_\beta\gamma_\lambda\not{p}'\gamma^\nu P_L+\gamma^\mu\not{p}'\gamma^\lambda\gamma_\alpha\gamma^\nu P_L\gamma_\beta\gamma_\lambda,
\end{aligned} \tag{4.12}$$

with $p' = p - q$. The Dirac algebra from eq. (4.12) is dramatically simplified by using heavy quark symmetry. Spinors $u_b(p)$ correspond to operators $b = h_b^{(v)} + \mathcal{O}(\frac{1}{m_b})$, so to leading order in $\frac{\Lambda_{QCD}}{m_b}$, we have

$$\langle B|\, \overline{u}_b(p)\Gamma u_b(p)\,|B\rangle = \frac{1}{2}\,\mathrm{Tr}[\Gamma\frac{1+\not{v}}{2}]. \tag{4.13}$$

The same formula applies to B^* mesons if we average over spin. Higher order corrections in $\frac{\Lambda_{QCD}}{m_b}$ can be computed by inserting heavy quark symmetry breaking terms from the heavy quark Lagrangian [12].

The integrals $C^{[\cdot,\alpha,\alpha\beta]}$ may be expressed, by means of the usual Feynman parametrization, momentum shifting, and integration, in terms of the Feynman parameter integral

$$S(a,b,c,d,e,f) = \int_0^1 dx \int_0^{1-x} dy \ln X(x,y;\mathbf{a}), \tag{4.14}$$

where

$$X(x,y;\mathbf{a}) = \frac{1}{\mu^2}(ax + by + cx^2 + dy^2 + exy + f),$$

and

$$\begin{aligned} a &= -(p^2 - m_b^2 + \lambda^2) \\ b &= m_j^2 - \lambda^2 - (p-q)^2 \\ c &= p^2 \\ d &= (p-q)^2 \\ e &= 2p\cdot(p-q) \\ f &= \lambda^2 - \mathrm{i}\epsilon. \end{aligned} \tag{4.15}$$

The real and imaginary parts of this function are evaluated analytically in Appendices A and B. The special case of a massless final state quark, which would give spurious divergences if the zero mass were substituted in the formulas in the previous appendices, is considered in Appendix C. These closed expressions are exact, and have been verified numerically. However, they are cumbersome to handle, and it may be simpler to evaluate these functions numerically than analytically. Expressions for the C integrals in terms of S and its derivatives are given in appendix D.

After evaluating the traces and integrals in terms of S and its derivatives, we arrive at the following results:

$$
\begin{aligned}
T_1^{vert} =& \frac{-\alpha_s}{3\pi}\,\frac{2}{\Delta_0}\;[(-m_b + \mathrm{v}\cdot\mathrm{q})\,(\frac{1}{\epsilon} - \frac{\gamma}{2} + \frac{1}{2}\ln 4\pi - 1 -\, S) \\
&+ m_b m_j^2 S_{,f} - m_b^3\,(2S_{,a} + 2S_{,b} - S_{,c} - S_{,d} - 2S_{,e} - S_{,f}) \\
&+\, m_b^2 \mathrm{v}\cdot\mathrm{q}\,(3S_{,a} + 5S_{,b} - S_{,c} - 3S_{,d} - 4S_{,e} - 2S_{,f}) \\
&+\, m_b q^2\,(S_{,a} + S_{,d} - S_{,f})\,+\,q^2\,\mathrm{v}\cdot\mathrm{q}\,(S_{,b} - S_{,d}) \\
&-\,2m_b(\mathrm{v}\cdot\mathrm{q})^2\,(S_{,a} + 2S_{,b} - S_{,d} - S_{,e} - S_{,f})] \equiv \frac{\alpha_s}{3\pi\Delta_0}\hat{T}_1^{vert} \\
T_2^{vert} =& \frac{\alpha_s}{3\pi}\,\frac{4}{\Delta_0}\;[m_b\,(\frac{1}{\epsilon} - \frac{\gamma}{2} + \frac{1}{2}\ln 4\pi - 1 -\, S) \\
&+\,(m_b^3 + m_b m_j^2 - m_b q^2)\,(S_{,a} + S_{,b} - S_{,f})\,+ m_b q^2 S_{,e})] \equiv \frac{\alpha_s}{3\pi\Delta_0}\hat{T}_2^{vert} \\
T_3^{vert} =& \frac{\alpha_s}{3\pi}\,\frac{2}{\Delta_0}\;[(\frac{1}{\epsilon} - \frac{\gamma}{2} + \frac{1}{2}\ln 4\pi - 1 -\, S) \\
&+\, m_b^2\,(3S_{,a} + 3S_{,b} - S_{,c} - S_{,d} - 2S_{,e} - 2S_{,f}) \\
&+\, q^2\,(S_{,b} - S_{,d})\,-\,2m_b\mathrm{v}\cdot\mathrm{q}\,(S_{,a} + 2S_{,b} - S_{,d} - S_{,e} - S_{,f})] \equiv \frac{\alpha_s}{3\pi\Delta_0}\hat{T}_3^{vert} \\
T_4^{vert} =& \frac{-\alpha_s}{3\pi}\,\frac{4}{\Delta_0}\,m_b\,(S_{,a} - S_{,e}) \equiv \frac{\alpha_s}{3\pi\Delta_0}\hat{T}_4^{vert} \\
T_5^{vert} =& \frac{-\alpha_s}{3\pi}\,\frac{2}{\Delta_0}\;[(\frac{1}{\epsilon} - \frac{\gamma}{2} + \frac{1}{2}\ln 4\pi - 1 -\, S) \\
&+\, m_b^2\,(S_{,a} + 2S_{,b} - 2S_{,f})\,+ m_j^2 S_{,b} \\
&-\,2\,m_b\mathrm{v}\cdot\mathrm{q}\,(S_{,a} + S_{,b} - S_{,e} - S_{,f})] \equiv \frac{\alpha_s}{3\pi\Delta_0}\hat{T}_5^{vert}.
\end{aligned}
\tag{4.16}
$$

The notation $S_{,a}$ means the derivative of $S(a,b,c,d,e,f)$ with respect to a. Again, we have two contributions to $Im(T^{vert})$: one from the imaginary part of the light quark propagator and one from the imaginary part of S and its derivatives. The former corresponds to the virtual gluon corrections to the vertex, and the latter to bremsstrahlung due to interference between initial and final state radiation.

4.4. Box Diagram

The box graph, depicted in fig. 5, gives

$$
\begin{aligned}
&\frac{4g^2}{3}\overline{u}_b(p)\int\frac{\mathrm{d}^D\,l}{(2\pi)^D}\frac{\gamma^\alpha(\not{l} + m_b)\gamma^\mu P_L(\not{l} - \not{q})\gamma^\nu(\not{l} + m_b)\gamma_\alpha}{[(l-p)^2 - \lambda^2 + \mathrm{i}\epsilon][(l-q)^2 - m_j^2 + \mathrm{i}\epsilon][l^2 - m_b^2 + \mathrm{i}\epsilon]^2}u_b(p) \\
&\qquad = \frac{\mathrm{i}\alpha}{3\pi}\left[N^{(B)}D + N^{(B)}_\alpha D^\alpha + N^{(B)}_{\alpha\beta}D^{\alpha\beta} + N^{(B)}_{\alpha\beta\gamma}D^{\alpha\beta\gamma}\right]u_b(p),
\end{aligned}
\tag{4.17}
$$

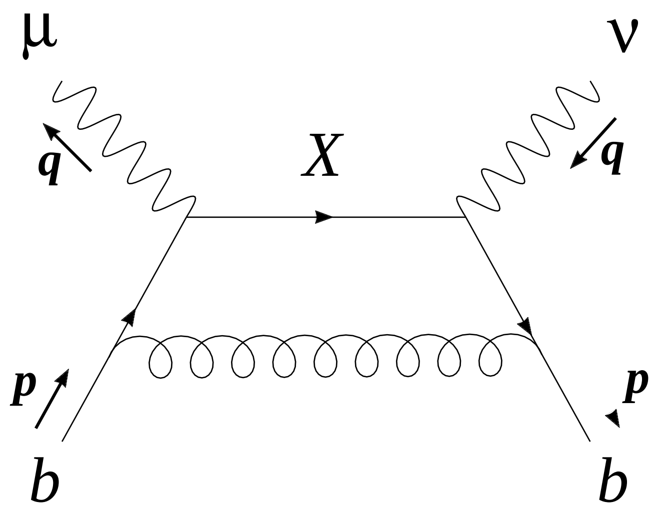

Fig. 5 Box diagram.

where

$$D^{[.,\alpha,\alpha\beta,\alpha\beta\gamma]} = \frac{-\mathrm{i}}{\pi^2}\int \frac{2\pi^{D-4}d^D l \quad [1, l^\alpha, l^\alpha l^\beta, l^\alpha l^\beta l^\gamma]}{[l^2 - m_b^2 + \mathrm{i}\epsilon]^2[(l-q)^2 - m_j^2 + \mathrm{i}\epsilon][(l-p)^2 - \lambda^2 + \mathrm{i}\epsilon]} \tag{4.18}$$

and

$$\begin{aligned}
N^{(B)} &= -m_b^2 \gamma^\lambda \gamma^\mu P_L \not{q} \gamma^\nu \gamma_\lambda \\
N_\alpha^{(B)} &= m_b^2 \gamma^\lambda \gamma^\mu P_L \gamma_\alpha \gamma^\nu \gamma_\lambda \\
&\quad - m_b \gamma^\lambda \gamma_\alpha \gamma^\mu P_L \not{q} \gamma^\nu \gamma_\lambda - m_b \gamma^\lambda \gamma^\mu P_L \not{q} \gamma^\nu \gamma_\alpha \gamma_\lambda \\
N_{\alpha\beta}^{(B)} &= m_b \gamma^\lambda \gamma_\alpha \gamma^\mu P_L \gamma_\beta \gamma^\nu \gamma_\lambda \\
&\quad + m_b \gamma^\lambda \gamma^\mu P_L \gamma_\alpha \gamma^\nu \gamma_\beta \gamma_\lambda - \gamma^\lambda \gamma_\alpha \gamma^\mu P_L \not{q} \gamma^\nu \gamma_\beta \gamma_\lambda \\
N_{\alpha\beta\gamma}^{(B)} &= \gamma^\lambda \gamma_\alpha \gamma^\mu P_L \gamma_\beta \gamma^\nu \gamma_\gamma \gamma_\lambda .
\end{aligned} \tag{4.19}$$

Evaluating the traces and integrals as with the vertex diagrams yields

$$
\begin{aligned}
T_1^{box} =& \frac{-\alpha_s}{3\pi}\,[m_b(3S_{,a}+S_{,c}+S_{,e})+m_b^3(-4S_{,a,a}-4S_{,a,b}+S_{,a,c}\\
&+S_{,a,d}+2S_{,a,e}+2S_{,a,f}+S_{,c,c}+3S_{,c,d}+3S_{,c,e}+S_{,d,e})\\
&+m_b q^2(-2S_{,a,b}+S_{,a,d}-2S_{,a,e}+S_{,c,d}+S_{,d,e})-\mathrm{v}\cdot\mathrm{q}\;(S_{,a}+S_{,e})\\
&+m_b^2\mathrm{v}\cdot\mathrm{q}\;(2S_{,a,a}+6S_{,a,b}+S_{,a,c}-S_{,a,d}-2S_{,a,f}-6S_{,c,d}-3S_{,c,e}-3S_{,d,e})\\
&+q^2\mathrm{v}\cdot\mathrm{q}\,(S_{,a,d}-S_{,d,e})+m_b(\mathrm{v}\cdot\mathrm{q})^2(-2S_{,a,d}+2S_{,c,d}+2S_{,d,e})]\equiv\frac{\alpha_s}{3\pi}\hat{T}_1^{box}\\
T_2^{box} =& \frac{\alpha_s}{3\pi}\;2\,[-m_b(S_{,a}-S_{,c}-S_{,e})\\
&+m_b^3(4S_{,a,a}+4S_{,a,b}-S_{,a,c}-S_{,a,d}-2S_{,a,e}-2S_{,a,f}-S_{,c,c}\\
&-3S_{,c,d}-3S_{,c,e}-S_{,d,e})-m_b q^2(S_{,a,d}-S_{,c,d}-S_{,d,e})]\equiv\frac{\alpha_s}{3\pi}\hat{T}_1^{box}\\
T_3^{box} =& \frac{\alpha_s}{3\pi}\;[-S_{,a}+3S_{,e}\\
&-m_b^2(-2S_{,a,a}-2S_{,a,b}-S_{,a,c}+S_{,a,d}+2S_{,a,f}-2S_{,c,d}-S_{,c,e}-S_{,d,e})\\
&-q^2(S_{,a,d}-S_{,d.e})-m_b\mathrm{v}\cdot\mathrm{q}\,(-2S_{,a,d}+2S_{,c,d}+2S_{,d,e})]\equiv\frac{\alpha_s}{3\pi}\hat{T}_1^{box}\\
T_4^{box} =& \frac{-\alpha_s}{3\pi}\;4\,[-m_b(S_{,a,b}+S_{,a,e}-S_{,c,d}-S_{,d,e})+\mathrm{v}\cdot\mathrm{q}\,(S_{,a,d}-S_{,d,e})]\equiv\frac{\alpha_s}{3\pi}\hat{T}_1^{box}\\
T_5^{box} =& \frac{-\alpha_s}{3\pi}\;[-S_{,a}+S_{,e}\\
&-m_b^2(-2S_{,a,a}-6S_{,a,b}-S_{,a,c}+S_{,a,d}+2S_{,a,f}+6S_{,c,d}+3S_{,c,e}+3S_{,d,e})\\
&-q^2(S_{,a,d}-S_{,d.e})-m_b\mathrm{v}\cdot\mathrm{q}\,(2S_{,a,d}-2S_{,c,d}-2S_{,d,e})]\equiv\frac{\alpha_s}{3\pi}\hat{T}_1^{box}.
\end{aligned}
\tag{4.20}
$$

The function $S_{,a,b}$ denotes the second derivative of $S(a,b,c,d,e,f)$ with respect to a and b. There is only one way to obtain an imaginary part from the T^{box}, via the imaginary part of S and its derivatives. This corresponds to cutting through the loop to obtain the bremsstrahlung due to initial state radiation.

5. Analytic Structure of Form Factors

The form factors $W_i(\mathrm{v}\cdot\mathrm{q},q^2)$ are given by the discontinuity of $T_i(\mathrm{v}\cdot\mathrm{q},q^2)$ across the cut in the complex $\mathrm{v}\cdot\mathrm{q}$ plane. The general location of the physical cuts may be deduced from the expression (1.6) by inserting a complete sets of states. There is a cut along the real axis for

$$\mathrm{v}\cdot\mathrm{q}\le\frac{m_B^2+q^2-m_X^2}{2m_B}. \tag{5.1}$$

This cut corresponds to weak amplitudes for which the initial hadronic state is a B (containing one b quark and one light antiquark), and the final state contains one j type quark (u or c) and the same light antiquark. m_X is the mass of the lightest such hadronic final state. The weak decay $B \to X\tau\overline{\nu}_\tau$ corresponds to the portion of this cut for which $\mathrm{v}\cdot\mathrm{q} > \sqrt{q^2}$ (since physical final state lepton pairs have $|\vec{q}|^2 \geq 0$). By crossing symmetry one obtains cuts in other portions of the above region, corresponding to different physical processes, such as $B\nu_\tau \to X\tau$.

There is one other class of cuts arising from the $x^0 < 0$ portion of the integral in equation (1.6), and corresponding to the production of two b quarks in the final state. This cut lies along the real axis for

$$\mathrm{v}\cdot\mathrm{q} \geq \frac{m_X'^2 - m_B^2 - q^2}{2m_B}, \tag{5.2}$$

where m_X' is the mass of the lightest hadronic state containing two b quarks which can be produced in a semileptonic weak process involving a B in the initial state. This process occurs only at order α_s^2.

The location of the branch cuts for the form factors T_i in the complex $\mathrm{v}\cdot\mathrm{q}$ plane for fixed q^2 can be explicitely determined by looking at the discontinuities of the T's along the real $\mathrm{v}\cdot\mathrm{q}$ axis. For the vertex and box diagrams, this means looking for an imaginary part of $S(\mathbf{a})$. $Im\,[S(\mathbf{a})]$ is simply the area within the triangular integration region over which the argument $X(x, y; \mathbf{a})$ is negative, the branch cut is determined by studying the curve $X(x, y; \mathbf{a}) = 0$. After expressing $\mathbf{a}$ in terms of physical variables, it is easy to show that $Im\,[S(\mathbf{a})] \neq 0$ if and only if $(p-q)^2 > m_j^2$. This implies a branch cut for $\mathrm{v}\cdot\mathrm{q} < \frac{m_b^2+q^2-m_j^2}{2m_b}$, in agreement with ref [2]. For the final quark self-energy, we have to look for an imaginary part in $\mathcal{A}\left[(p-q)^2, m_j^2\right]$, which leads us to the same cut.

An alternate way to see this is to consider two particle unitarity. Unitarity of the amplitude A_{if} between initial state $|i\rangle$ and final state $\langle f|$ implies

$$A_{fi} - A_{if}^* = \sum_n \mathrm{i}(2\pi)^4\delta^{(4)}(p_i - p_n)A_{fn}A_{in}^*.$$

For a two particle intermediate state (in our case a gluon and a quark), the right hand side becomes

$$\frac{i}{2}\frac{k}{16\pi^2 E_T}\int d\Omega\,\langle f|\,A\,|k_1, k_2\rangle\,\langle k_1, k_2|\,A^\dagger\,|i\rangle\,\theta(E_T^2 - m_j^2),$$

where k is the magnitude of the three-momentum of either intermediate particle in the center of mass frame, and E_T is total center of mass energy. Since

$$\frac{k}{E_T} = \frac{1}{2}\sqrt{(1 - \frac{m_j^2}{(p-q)^2})^2},$$

two particle unitarity provides a positive check on our calculation.

6. Divergence Cancellation

6.1. Ultraviolet Divergences

We performed the loop momentum integration in $4-\epsilon$ dimensions in order to regulate the ultraviolet divergences. The divergent part of each graph is always proportional to the factor

$$\frac{2}{\epsilon} - \gamma + \ln(4\pi\mu^2). \tag{6.1}$$

The final and initial quark self-energies contribute the same ultraviolet divergence, while the box is convergent. One can see from (4.4) and (4.16) that the divergent contribution from the vertex is -2 times that of the initial quark self-energy, so the dependence on the regulator ϵ and the renormalization scale μ cancel exactly. This is nothing but an explicit application of the Ward identity to our case.

6.2. Infrared Divergences

The cancellation of infrared divergences (which we have regulated with a gluon mass) is more involved. The triple differential decay distribution is IR divergent; the dependence on the gluon mass cancels only after integration over v$\cdot$q. The self-energies exhibit the IR divergence explicitely as $\ln\lambda^2$ terms, but in the vertex and box diagrams the divergences are hidden in the S functions. It can be seen by numerical evaluation that of all the S functions, only $S_{,f}$ and $S_{,a,f}$ blow up for zero gluon mass. These two functions can be written as a gluon independent term plus another term proportional to $\ln\lambda^2$. Upon integration over v$\cdot$q, we have verified that the coefficients of these logarithms cancel exactly the IR logarithms from the self-energy. This non trivial cancellation of the gluon mass dependence give us confidence in our formulas. We have numerically verified this cancellation for up and charm final quark masses and for several values of q^2. The cancellation is independent of the tau energy, since all the T form factors are independent of E_τ.

7. Λ_b baryon decay

We now calculate the additional spin-dependent form factors G_1 through G_9 that describe the decay of a polarized Λ_b. The Dirac algebra is a little more complicated in this case since we cannot use equation eq. (2.9). First we reduce the expressions given in eqs. (4.12) and (4.19) by means of the Chisholm identity

$$\gamma^\mu\gamma^\alpha\gamma^\nu = g^{\mu\alpha}\gamma^\nu + g^{\nu\alpha}\gamma^\mu - g^{\mu\nu}\gamma^\alpha + i\epsilon^{\mu\nu\alpha\beta}\gamma_\beta\gamma_5. \tag{7.1}$$

Note that in our convention $\epsilon_{0123} = +1$. After some algebra which we performed with the help of Feyncalc [13] the Dirac structure of the N's reduces to terms with zero or one gamma matrix and γ_5's. The matrix elements of these operators can be further simplified using the heavy quark symmetry. To first order in $1/m_b$ we have

$$\begin{aligned}
\langle\Lambda_b(v,s)|\,\overline{h}_b\Gamma h_b\,|\Lambda_b(v,s)\rangle &= \overline{u}(v,s)\Gamma u(v,s),\\
\overline{u}(v,s)u(v,s) &= 1,\\
\overline{u}(v,s)\gamma_5 u(v,s) &= 0,\\
\overline{u}(v,s)\gamma^\lambda u(v,s) &= v^\lambda,\\
\overline{u}(v,s)\gamma^\lambda\gamma_5 u(v,s) &= s^\lambda.
\end{aligned} \tag{7.2}$$

We now contract the resulting expressions with the C's given in Appendix D and collect terms with respect to their Lorentz structure. The baryon form factors T_1 to T_5, which enter into the first line of equation (3.9), are identical to the meson form factors T_1 to T_5 in equations (4.4), (4.5), (4.16), and (4.20). The spin-dependent form factors in the rest of (3.9) are found by comparing the above result to the defining equation for the G's, eq. (3.8).

We divide the spin dependent form factors into three pieces

$$G_i = G_i^0 + G_i^{vert} + G_i^{box}.$$

The results from wavefunction renormalization and the self-energy graph are proportional to the tree level result, so they may be combined into

$$\begin{aligned}
G_1^0 &= \frac{-1}{2\Delta_0}(1+\delta Z_b + \frac{\alpha_s}{3\pi\Delta_0}\mathcal{A}[(p-q)^2, m_j^2])\\
G_6^0 &= \frac{-m_b}{2\Delta_0}(1+\delta Z_b + \frac{\alpha_s}{3\pi\Delta_0}\mathcal{A}[(p-q)^2, m_j^2])\\
G_7^0 &= \frac{1}{2\Delta_0}(1+\delta Z_b + \frac{\alpha_s}{3\pi\Delta_0}\mathcal{A}[(p-q)^2, m_j^2])\\
G_8^0 &= \frac{m_b}{2\Delta_0}(1+\delta Z_b + \frac{\alpha_s}{3\pi\Delta_0}\mathcal{A}[(p-q)^2, m_j^2])\\
G_9^0 &= \frac{-1}{2\Delta_0}(1+\delta Z_b + \frac{\alpha_s}{3\pi\Delta_0}\mathcal{A}[(p-q)^2, m_j^2])\\
G_2^0 &= G_3^0 = G_4^0 = G_5^0 = 0,
\end{aligned} \tag{7.3}$$

where the functions D^0 and $\mathcal{A}$ are defined in eq.s (4.4) and (4.6).

The contributions to the G's from the vertex diagrams are

$$
\begin{aligned}
G_1^{vert} =& \frac{\alpha}{3\pi}\frac{2}{\Delta_0}(-(\frac{1}{\epsilon}-\frac{\gamma}{2}+\frac{1}{2}\ln(4\pi))+1+S-m_b^2(3S_{,a}+3S_{,b}-S_{,c}-S_{,d} \\
&-2S_{,e}-2S_{,f})-q^2(S_{,b}-S_{,d})+2m_b\mathrm{v\cdot q}(S_{,a}+2S_{,b}-S_{,d}-S_{,e}-S_{,f})) \\
G_2^{vert} =& \frac{\alpha}{3\pi}\frac{4m_b^2}{\Delta_0}(-S_{,a}+S_{,c}+S_{,e}) \\
G_3^{vert} =& \frac{\alpha}{3\pi}\frac{m_b}{\Delta_0}S_{,e} \\
G_4^{vert} =& 0 \\
G_5^{vert} =& \frac{\alpha}{3\pi}\frac{2m_b}{\Delta_0}(S_{,a}-S_{,e}) \\
G_6^{vert} =& \frac{\alpha}{3\pi}\frac{2m_b}{\Delta_0}(-(\frac{1}{\epsilon}-\frac{\gamma}{2}+\frac{1}{2}\ln(4\pi))+1+S-m_j^2(S_{,a}+S_{,b}-S_{,f}) \\
&-m_b^2(S_{,a}+S_{,b}-S_{,f})+q^2(S_{,a}+S_{,b}-S_{,e}-S_{,f}) \\
&-m_b\mathrm{v\cdot q}(S_{,a}-S_{,c}-S_{,e})) \\
G_7^{vert} =& \frac{\alpha}{3\pi}\frac{2}{\Delta_0}(\frac{1}{\epsilon}-\frac{\gamma}{2}+\frac{1}{2}\ln(4\pi)-1-S+m_j^2S_{,b}-m_b\mathrm{v\cdot q}(S_{,a}+2S_{,b}-S_{,e}-2S_{,f}) \\
&+m_b^2(2S_{,a}+2S_{,b}-S_{,c}-S_{,e}-2S_{,f})) \\
G_8^{vert} =& \frac{\alpha}{3\pi}\frac{m_b}{\Delta_0}(2(\frac{1}{\epsilon}-\frac{\gamma}{2}+\frac{1}{2}\ln(4\pi))-2-2S-q^2(2S_{,a}+2S_{,d}+S_{,e}-2S_{,f}) \\
&-2m_j^2S_{,f}+2m_b^2(2S_{,a}+2S_{,b}-S_{,c}-S_{,d}-2S_{,e}-S_{,f}) \\
&-m_b\mathrm{v\cdot q}(3S_{,a}+6S_{,b}-S_{,c}-4S_{,d}-5S_{,e}-2S_{,f})) \\
G_9^{vert} =& \frac{\alpha}{3\pi}\frac{1}{\Delta_0}(-2(\frac{1}{\epsilon}-\frac{\gamma}{2}+\frac{1}{2}\ln(4\pi))+2+2S-2q^2(S_{,b}-S_{,d}) \\
&-m_b^2(3S_{,a}+4S_{,b}-S_{,c}-2S_{,d}-3S_{,e}-2S_{,f}) \\
&+m_b\mathrm{v\cdot q}(4S_{,a}+8S_{,b}-4S_{,d}-3S_{,e}-4S_{,f})).
\end{aligned}
\tag{7.4}
$$

The contribution to the G's from the box are calculated in the same way as the vertex contributions. Again, the algebra is a little tedious but straightforward. Making repeated use of the Chisholm identity eq. (7.1) , the heavy quark matrix elements eq. (7.2), and

contracting with the D's eq. (4.18) we arrive at

$$
\begin{aligned}
G_1^{box} =& \frac{\alpha}{3\pi}(S_{,a} + S_{,e} - q^2(S_{,a,d} - S_{,d,e}) + 2m_b \mathrm{v}\cdot\mathrm{q}(S_{,a,d} - S_{,c,d} - S_{,d,e}) \\
& - m_b^2(2S_{,a,a} + 2S_{,a,b} - S_{,a,c} + S_{,a,d} - 2S_{,a,f} - 2S_{,c,d} - S_{,c,e} - S_{,d,e})) \\
G_2^{box} =& \frac{\alpha}{3\pi} 4m_b^2(S_{,a,b} - 2S_{,a,d} - 2S_{,a,e} + 2S_{,c,d} + S_{,c,e} + S_{,d,e}) \\
G_3^{box} =& \frac{\alpha}{3\pi} m_b(-S_{,a,b} + S_{,a,d} - S_{,a,e}) \\
G_4^{box} =& \frac{\alpha}{3\pi} 4(-S_{,a,d} + S_{,d,e}) \\
G_5^{box} =& \frac{\alpha}{3\pi} 2m_b(-S_{,a,b} + 3S_{,a,d} + S_{,a,e} - 2S_{,c,d} - 2S_{,d,e}) \\
G_6^{box} =& \frac{\alpha}{3\pi}(m_b^3(-4S_{,a,a} - 4S_{,a,b} + 3S_{,a,c} + 3S_{,a,d} + 6S_{,a,e} + 2S_{,a,f} - S_{,c,c} \\
& - 3S_{,c,d} - 3S_{,c,e} - S_{,d,e}) + m_b(S_{,a} - S_{,c} - S_{,e}) + m_b q^2(S_{,a,d} - S_{,c,d} - S_{,d,e}) \\
& + 2m_b^2 \mathrm{v}\cdot\mathrm{q}(S_{,a,b} - 2S_{,a,d} - 2S_{,a,e} + 2S_{,c,d} + S_{,c,e} + S_{,d,e})) \\
G_7^{box} =& \frac{\alpha}{3\pi}(-S_{,a} + S_{,e} + q^2(-S_{,a,d} + S_{,d,e}) + m_b^2(2S_{,a,a} + 4S_{,a,b} \\
& - S_{,a,c} - 3S_{,a,d} - 4S_{,a,e} - 2S_{,a,f} + 2S_{,c,d} + S_{,c,e} + S_{,d,e}) \\
& - 2m_b(S_{,a,b} - 2S_{,a,d} - S_{,a,e} + S_{,c,d} + S_{,d,e})) \\
G_8^{box} =& \frac{\alpha}{3\pi}(m_b^3(4S_{,a,a} + 4S_{,a,b} - 3S_{,a,c} - 3S_{,a,d} - 6S_{,a,e} - 2S_{,a,f} + S_{,c,c} + 3S_{,c,d} \\
& + 3S_{,c,e} + S_{,d,e}) + m_b q^2(3S_{,a,b} - 4S_{,a,d} - S_{,a,e} + S_{,c,d} + S_{,d,e}) \\
& - 3m_b(S_{,a} - S_{,c} - S_{,e}) - m_b^2 \mathrm{v}\cdot\mathrm{q}(3S_{,a,a} + 7S_{,a,b} - S_{,a,c} - 7S_{,a,d} - 8S_{,a,e} \\
& - 2S_{,a,f} + 4S_{,c,d} + 2S_{,c,e} + 2S_{,d,e})) \\
G_9^{box} =& \frac{\alpha}{3\pi}(S_{,a} - 3S_{,e} + q^2(S_{,a,d} - S_{,d,e}) - m_b \mathrm{v}\cdot\mathrm{q}(S_{,a,b} + S_{,a,d} + S_{,a,e} - 2S_{,c,d} \\
& - 2S_{,d,e}) + m_b^2(S_{,a,a} + S_{,a,b} - 2S_{,c,d} - S_{,c,e} - S_{,d,e})).
\end{aligned}
\tag{7.5}
$$

The self-energy diagram gives no spin dependent contribution.

8. Conclusions

We have presented α_s corrections to triple differential inclusive distributions for $B \to X_q \tau \overline{\nu}$, in closed analytic form. We have also calculated the dependence of the distribution on the tau polarization, for both up and charm quarks. We have not performed the integration over neutrino energy which is necessary to give the physically meaningful double differential distribution, but have instead left the answer in terms of a one dimensional

integral which can be evaluated numerically. An additional integration over tau energy or momentum transfer will yield $\frac{d\Gamma}{dq^2}$ or $\frac{d\Gamma}{dE_\tau}$, respectively. Formulas appropriate for insertion in a Monte Carlo program are presented in Appendix E.

We have verified numerically the cancellation of infrared divergences due to soft gluons, and the stability of the double differential distribution against variations in the infared regulating gluon mass. In addition, the formulas for the graphs and integrals entering the triple differential distribution have been checked extensively numerically.

Our computation has been in the context of a heavy quark operator product expansion. An advantage of this is that extending the calculation to polarized $\Lambda' s$ is simple. The quartic distribution for polarized Lambda's to polarized tau's $+\overline{\nu} + X_q$ involves nine additional form factors, which we have computed. Again, integration over the neutrino energy is necessary to get an experimentally observable distribution.

The formulas in this paper may be useful for purely theoretical reasons, in addition to experimental ones. A computation of the α_s corrections to the Wilson coefficeints of the operator product expansion may give insight on how perturbation theory breaks down as one nears the endpoint of the lepton spectrum. This requires calculating one-loop graphs in QCD (which we have done) and comparing them to one-loop graphs in the effective theory.

Strong corrections to form factors at $\mathcal{O}(\frac{1}{m_b^2})$ may also be of interest [14]. The structure of our results is amenable to such a calculation, by expressing the b momentum as $p = m_b v + k$ and expanding in residual momentum k. The only complication here is that care must be taken not to cut the b quark lines when computing Bremstrahlung; this requires a modification of our formula for the imaginary part of S.

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Appendix A: Evaluation of Re(S)

We want to evaluate

$$S(a,b,c,d,e,f) = \int_0^1 dx \int_0^{1-x} dy \ln X(x,y;\mathbf{a})$$

where

$$X(x,y;\mathbf{a}) = \frac{1}{\mu}(ax + by + cx^2 + dy^2 + exy + f).$$

We evaluate the real and imaginary parts of S separately because we will use a change of variables which causes our integration contour to cross branch cuts for some values of our parameters, producing spurious contributions of iπ. In addition, this change of variables is valid only if $d \neq 0$. For an on-shell up quark, we may either take the limit as $d \to 0$, or use the formula in appendix C.

First, we perform a change of variables, $x \to 1 - x$ and $y \to y - \alpha x$, where

$$\alpha = \frac{1}{2d}(e + \sqrt{e^2 - 4cd}),$$

and use the identity

$$\int_0^1 dx \int_{-\alpha x}^{(1-\alpha)x} dy = \int_0^{1-\alpha} dy \int_{\frac{y}{(1-\alpha)}}^1 dx - \int_0^{-\alpha} dy \int_{-\frac{y}{\alpha}}^1 dx.$$

In the first integral on the right hand side, make the replacement $y \to (1-\alpha)y$, and in the second, replace $y \to -\frac{y}{\alpha}$. The resulting equality is valid only for the real parts:

$$Re[S(\mathbf{a})] = Re\{\int_0^1 dy \int_y^1 dx \, [(1-\alpha)\ln X(\mathbf{a}_1) + \alpha \ln X(\mathbf{a}_2)]\},$$

where

$$\begin{aligned}
&\mathbf{a}_i = \{a_i, b_i, c_i, d_i, e_i, f_i\}, \\
&a_1 = \alpha(b+e) - a - 2c = a_2, \\
&b_1 = (1-\alpha)(b+e), \qquad b_2 = -\alpha(b+e), \\
&c_1 = 0 = c_2, \\
&d_1 = (1-\alpha)^2 d, \qquad d_2 = \alpha^2 d, \\
&e_1 = (1-\alpha)(2d\alpha - e), \qquad e_2 = \alpha(e - 2d\alpha), \\
&f_1 = f + a + c = f_2.
\end{aligned} \tag{A.1}$$

This expression can be evaluated straightforwardly by completing the squares inside each integral. The final result is

$$Re[S(\mathbf{a})] = Re[(1-\alpha)F(\mathbf{a}_1) + \alpha F(\mathbf{a}_2)] \tag{A.2}$$

where

$$F(\mathbf{a}) = \frac{1}{e}\left[-f(\mathbf{a}_3) + f(\mathbf{a}_4)\right],$$

$$f(\mathbf{a}) = a\{g(a,b,c,d) + g(a,-b,c,d) - 1 + 3c - 2d$$
$$+ (\frac{1}{2} - c + d)\ln a + (c^2 - b^2)(1 - \ln a)\,[\ln(c+d) - \ln(1+c+d)]\}, \tag{A.3}$$

$$g(a,b,c,d) = \ln(b+d)\left[\frac{1}{2}b^2 + bc + cd - \frac{1}{2}d^2 + (b^2 - c^2)\ln(\frac{c+d}{c-b})\right]$$
$$+ \ln(1+b+d)\left[\frac{1}{2} - \frac{1}{2}b^2 - c - bc + d - cd + \frac{1}{2}d^2 + (c^2 - b^2)\ln(\frac{1+c+d}{c-b})\right] \tag{A.4}$$
$$+ (c^2 - b^2)\left[Sp(\frac{1+b+d}{b-c}) - Sp(\frac{b+d}{b-c})\right],$$

$$\begin{aligned}
&a_3 = d + e, \qquad a_4 = d,\\
&b_3 = \sqrt{(\frac{a+b}{2d+2e})^2 - \frac{f}{d+e}}, \qquad b_4 = \sqrt{(\frac{b+e}{2d})^2 - \frac{a+f}{d}},\\
&c_3 = \frac{a}{e} - \frac{a+b}{2d+2e}, \qquad c_4 = \frac{a}{e} - \frac{b+e}{2d},\\
&d_3 = \frac{a+b}{2d+2e}, \qquad d_4 = \frac{b+e}{2d}\\
&e_3 = e_4 = f_3 = f_4 = 0,
\end{aligned} \tag{A.5}$$

and the Spence function is defined so that

$$Sp(x) = \int_x^0 \frac{\ln(1-t)}{t}dt.$$

Note that in the evaluation of, for example, $F(\mathbf{a}_1)$, the parameters which enter into the definitions of $\mathbf{a}_3$ and $\mathbf{a}_4$ in equation (A.5) are $\mathbf{a}_1$.

Appendix B: Evaluation of Im(S)

It is much easier to evaluate the imaginary part of S than S itself.

$$ImS = \int_0^1 dx \int_0^{1-x} dy \; Im \ln X(x,y;\mathbf{a})$$

is just π times the area in which X is negative, inside the triangle in the (x,y) plane defined by the integration limits (recall that $f = \lambda^2 - i\epsilon$ is the only complex input parameter). To obtain this area, we solve the equation $X = 0$, which gives us a hyperbola in the (x,y)

plane, and integrate x as a function of y (or viceversa) in the triangle. The evaluation of this area gives:

$$ImS = \pi(A(\sqrt{b^2 - 4\,d\,f}) - A(-\sqrt{b^2 - 4\,d\,f})),$$

where the function A is

$$\begin{aligned} A(K) =& \frac{a\ (b-K)}{4\,c\,d} - \frac{e\,(b-K)^2}{16\,c\,d^2} + \left(\frac{2\,b\,c - a\,e}{-4\,c\,(e^2 - 4\,c\,d)} + \frac{-b+K}{8\,c\,d}\right) \times \\ & \sqrt{\frac{2\,a^2\,d^2 - 2\,a\,b\,d\,e + b^2\,e^2 - 2\,d\,e^2\,f + 2\,a\,d\,e\,K - b\,e^2\,K}{2d^2}} \\ & + \frac{\left(a\,b\,e - b^2\,c - a^2\,d - f(e^2 - 4\,c\,d)\right)}{(e^2 - 4\,c\,d)^{\frac{3}{2}}}\ \ln[-4\,b\,c + 2\,a\,e - \frac{-(e^2 - 4\,c\,d)\ (b-K)}{d} + \\ & \sqrt{e^2 - 4\,c\,d}\sqrt{\frac{4\,a^2\,d^2 + 2\,(2\,a\,d\,e - b\,e^2)(K-b) - 4\,d\,e^2\,f}{d^2}}]. \end{aligned} \tag{B.1}$$

All the derivatives of S needed for the decay rate can be obtained from this formula.

Appendix C: Evaluation of Re(S) at threshold, for massless final quarks

The formula for the real part of S breaks down for $d = 0$, and it is cumbersome to take the limit numerically. It is convenient to have available a simpler representation of S which is valid when $d = 0$. The evaluation is straightforward, so we omit details and present only the final result:

$$\begin{aligned} Re[S(a,b,c,0,e,f)] = Re[&-\frac{1}{2} + \frac{1}{2}\ln(c-e) \\ &- \frac{c}{e}\ln(c)[\frac{1}{2} + 2a_1 - \frac{b}{e} + (c_1^2 + (-a_1 + \frac{b}{e})^2)\ln(\frac{b}{b+e})] \\ &+ \frac{1}{e}\ln(c-e)[\frac{c}{2} + a_2 c + t_3 + t_4 \ln(\frac{1 + a_2 + t_2}{a_2 + t_2})] \\ &+ \frac{c}{e} f_0(a_2, \frac{t_3}{c}, c_2) - \frac{c}{e} f_0(a_1, a_1 - \frac{b}{e}, c_1) - f_0(a_2, -1 - a_2, c_2) \\ &+ t_1 g_0(a_1, -a_1 + \frac{b}{e}, c_1) + \frac{t_4}{e} g_0(a_2, t_2, c_2)] \end{aligned} \tag{C.1}$$

where

$$f_0(a,b,c) = f_{00}(a,b,c) + f_{00}(a,b,-c) - \frac{1}{2} - a - 2b,$$

$$g_0(a,b,c) = g_{00}(a,b,c) + g_{00}(a,b,-c),$$

$$\begin{aligned} f_{00}(a,b,c) = &(-\frac{1}{2}a^2 - ab - bc + \frac{1}{2}c^2)\ln(a+c) \\ &+ (\frac{1}{2} + a + \frac{1}{2}a^2 + b + ab + bc - \frac{1}{2}c^2)\ln(1 + a + c), \end{aligned} \tag{C.2}$$

$$g_{00}(a,b,c) = \ln(\frac{1+a+b}{b-c})\ln(1+a+c) - \ln(\frac{a+b}{b-c})\ln(a+c) \\ + Sp(\frac{1+a+c}{c-b}) - Sp(\frac{a+c}{c-b}), \tag{C.3}$$

and

$$\begin{aligned}
a_1 &= \frac{a}{2c} \\
a_2 &= \frac{a-b+e}{2(c-e)} \\
c_1 &= \sqrt{\frac{a^2}{4c^2} - \frac{f}{c}} \\
c_2 &= \sqrt{\frac{(a-b+e)^2}{4(c-e)^2} - \frac{b+f}{c-e}} \\
t_1 &= \frac{c}{e}[c_1^2 - (\frac{b}{e} - a_1)^2] \\
t_2 &= \frac{b}{e} - a_2 \\
t_3 &= a - 2ca_2 - ct_2 \\
t_4 &= ct_2^2 + (2ca_2 - a)t_2 + f - aa_2 + ca_2^2.
\end{aligned} \tag{C.4}$$

The real part is taken because, as in Appendix A, the changes of variable contribute extraneous factors of iπ to the imaginary part of S. While both equations (A.2) and (C.1) are exact, single precision numeric evaluations of them may work poorly when the intermediate parameters are anomalously large or small (for example, when $c \approx e$ above).

Appendix D: The loop integrals in terms of S's

$$\begin{aligned}
C &= -S_{,f} \\
C^\alpha &= -S_{,b}\; q^\alpha + (S_{,a} + S_{,b} - S_{,f})\; p^\alpha \\
C^{\alpha\beta} &= (-\frac{1}{2}S + \frac{1}{2}(\frac{1}{\epsilon} - \frac{\gamma}{2} + \frac{1}{2}\ln 4\pi))\, g^{\alpha\beta} - S_{,d}\; q^\alpha\, q^\beta \\
&\quad - (S_{,b} - S_{,d} - S_{,e})\,\left(q^\beta\, p^\alpha + q^\alpha\, p^\beta\right) \\
&\quad + (2\,S_{,a} + 2\,S_{,b} - S_{,c} - S_{,d} - 2\,S_{,e} - S_{,f})\; p^\alpha\, p^\beta
\end{aligned} \tag{D.1}$$

$$D = -S_{,a,f}$$

$$D^{\alpha} = -S_{,a,b}\; q^{\alpha} + (S_{,a,a} + S_{,a,b} - S_{,a,f})\; p^{\alpha}$$

$$\begin{aligned} D^{\alpha\beta} = & -\frac{1}{2}\, S_{,a}\, g^{\alpha\beta} - S_{,a,d}\, q^{\alpha}\, q^{\beta} \\ & - (S_{,a,b} - S_{,a,d} - S_{,a,e})\, \left(q^{\beta}\, p^{\alpha} + q^{\alpha}\, p^{\beta} \right) \\ & + (2\, S_{,a,a} + 2\, S_{,a,b} - S_{,a,c} - S_{,a,d} - 2\, S_{,a,e} - S_{,a,f})\; p^{\alpha}\, p^{\beta} \end{aligned}$$

$$\begin{aligned} D^{\alpha\beta\gamma} = & -\frac{1}{2}\, S_{,e}\, \left(g^{\beta\gamma}\, q^{\alpha} + g^{\alpha\gamma}\, q^{\beta} + g^{\alpha\beta}\, q^{\gamma} \right) - S_{,d,e}\; q^{\alpha}\, q^{\beta}\, q^{\gamma} \\ & - \frac{1}{2}\, (S_{,a} - S_{,c} - S_{,e})\, \left(g^{\alpha\gamma}\, p^{\beta} + g^{\beta\gamma}\, p^{\alpha} + g^{\alpha\beta}\, p^{\gamma} \right) \\ & - (S_{,a,d} - S_{,c,d} - S_{,d,e})\, \left(q^{\alpha}\, q^{\gamma}\, p^{\beta} + q^{\beta}\, q^{\gamma}\, p^{\alpha} + q^{\alpha}\, q^{\beta}\, p^{\gamma} \right) \\ & - (S_{,a,b} - 2\, S_{,a,d} - 2\, S_{,a,e} + 2\, S_{,c,d} + S_{,c,e} + S_{,d,e}) \\ & \left(q^{\alpha}\, p^{\beta}\, p^{\gamma} + q^{\beta}\, p^{\alpha}\, p^{\gamma} + q^{\gamma}\, p^{\alpha}\, p^{\beta} \right) \\ & + (3\, S_{,a,a} + 3\, S_{,a,b} - 3\, S_{,a,c} - 3\, S_{,a,d} - 6\, S_{,a,e} \\ & \qquad - S_{,a,f} + S_{,c,c} + 3\, S_{,c,d} + 3\, S_{,c,e} + S_{,d,e})\, p^{\alpha}\, p^{\beta}\, p^{\gamma} \ . \end{aligned} \tag{D.2}$$

Appendix E: Double differential B meson Decay rate

We collect here the complete formula for the α_s correction to the double differential B meson decay rate. It is expressed as a one dimensional integral amenable to insertion into a numeric integration program incorporating any relevant cuts and detector efficiencies. The masses m_τ, m_b, m_j correspond to the tau lepton, the b quark, and the u or c quark masses, respectively. E_τ and E_ν are the tau and neutrino energies, v is the velocity of the B meson, and $\mathrm{v}\cdot\mathrm{q} = E_\tau + E_\nu$ in the center of mass frame.

The double differential rate is

$$
\begin{aligned}
\frac{d^2\Gamma}{dq^2 dE_\tau} &= \frac{-\alpha_s\,|V_{jb}|^2\,G_F^2}{6\pi^3}\Bigg[(q^2-m_\tau^2)\,\Big((m_b-\mathrm{v}\!\cdot\!\mathrm{q}_{th})(-4+3\ln m_b^2-2\ln\lambda^2)+\hat{T}_1^{vert}|_{th} \\
&- \int d\,\mathrm{v}\!\cdot\!\mathrm{q}\big(\frac{(m_b-\mathrm{v}\!\cdot\!\mathrm{q})\mathcal{B}[\mathrm{v}\!\cdot\!\mathrm{q},m_j^2]}{2(m_b^2+q^2-2m_b\mathrm{v}\!\cdot\!\mathrm{q}-m_j^2)^2}+\frac{Im(\hat{T}_1^{vert})/\pi}{(m_b^2+q^2-2m_b\mathrm{v}\!\cdot\!\mathrm{q}-m_j^2)}+\frac{Im(\hat{T}_1^{box})}{\pi}\big)\Big) \\
&+\left(2E_\tau(\mathrm{v}\!\cdot\!\mathrm{q}|_{th}-E_\tau)-\frac{q^2-m_\tau^2}{2}\right)\Big(2m_b(-4+3\ln m_b^2-2\ln\lambda^2)+\hat{T}_2^{vert}|_{th}\Big) \\
&- \int d\,\mathrm{v}\!\cdot\!\mathrm{q}\left(2E_\tau(\mathrm{v}\!\cdot\!\mathrm{q}-E_\tau)-\frac{q^2-m_\tau^2}{2}\right)\Big(\frac{m_b\,\mathcal{B}[\mathrm{v}\!\cdot\!\mathrm{q},m_j^2]}{(m_b^2+q^2-2m_b\mathrm{v}\!\cdot\!\mathrm{q}-m_j^2)^2} \\
&+\frac{Im(\hat{T}_2^{vert})/\pi}{(m_b^2+q^2-2m_b\mathrm{v}\!\cdot\!\mathrm{q}-m_j^2)}+\frac{Im(\hat{T}_2^{box})}{\pi}\Big) \\
&+\big((2E_\tau-\mathrm{v}\!\cdot\!\mathrm{q}|_{th})(q^2-m_\tau^2)-2m_\tau^2(\mathrm{v}\!\cdot\!\mathrm{q}|_{th}-E_\tau)\big)\Big((-4+3\ln m_b^2-2\ln\lambda^2)+\hat{T}_3^{vert}|_{th}\Big) \\
&- \int d\,\mathrm{v}\!\cdot\!\mathrm{q}\,\big((2E_\tau-\mathrm{v}\!\cdot\!\mathrm{q})(q^2-m_\tau^2)-2m_\tau^2(\mathrm{v}\!\cdot\!\mathrm{q}-E_\tau)\big)\big(\frac{\mathcal{B}[\mathrm{v}\!\cdot\!\mathrm{q},m_j^2]}{2(m_b^2+q^2-2m_b\mathrm{v}\!\cdot\!\mathrm{q}-m_j^2)^2} \\
&+\frac{Im(\hat{T}_3^{vert})/\pi}{(m_b^2+q^2-2m_b\mathrm{v}\!\cdot\!\mathrm{q}-m_j^2)}+\frac{Im(\hat{T}_3^{box})}{\pi}\big) \\
&+\left(\frac{q^2-m_\tau^2}{2}\right)m_\tau^2\Big(\hat{T}_4^{vert}|_{th}-\int d\,\mathrm{v}\!\cdot\!\mathrm{q}\big(\frac{Im(\hat{T}_4^{vert})/\pi}{(m_b^2+q^2-2m_b\mathrm{v}\!\cdot\!\mathrm{q}-m_j^2)}+\frac{Im(\hat{T}_4^{box})}{\pi}\big)\Big) \\
&+2m_\tau^2\Big((\mathrm{v}\!\cdot\!\mathrm{q}|_{th}-E_\nu)(-(-4+3\ln m_b^2-2\ln\lambda^2)+\hat{T}_5^{vert}|_{th}) \\
&- \int d\,\mathrm{v}\!\cdot\!\mathrm{q}\,(\mathrm{v}\!\cdot\!\mathrm{q}-E_\nu)\,\big(\frac{-\mathcal{B}[\mathrm{v}\!\cdot\!\mathrm{q},m_j^2]}{2(m_b^2+q^2-2m_b\mathrm{v}\!\cdot\!\mathrm{q}-m_j^2)^2} \\
&+\frac{Im(\hat{T}_5^{vert})/\pi}{(m_b^2+q^2-2m_b\mathrm{v}\!\cdot\!\mathrm{q}-m_j^2)}+\frac{Im(\hat{T}_5^{box})}{\pi}\big)\Big)\Bigg],
\end{aligned}
\tag{E.1}
$$

where

$$
\mathcal{B}[\mathrm{v}\!\cdot\!\mathrm{q},m_j^2]=\frac{m_j^6}{(m_b^2+q^2-2m_b\mathrm{v}\!\cdot\!\mathrm{q})^2}-7\frac{m_j^4}{(m_b^2+q^2-2m_b\mathrm{v}\!\cdot\!\mathrm{q})}-m_b^2-q^2+2m_b\mathrm{v}\!\cdot\!\mathrm{q}+7m_j^2.
$$

The $\hat{T}|_{th}$ functions are the real parts of the $\hat{T}$ functions defined in (4.16) and (4.20), evaluated at the parton model theshold $\mathrm{v}\!\cdot\!\mathrm{q}|_{th}=(m_b^2+q^2-m_j^2)/2m_b$. The real and imaginary parts of $\hat{T}$ may be expressed in terms of the real part of the function S evaluated in Appendices A and C, and the imaginary part of S evaluated in Appendix B. Integration limits are given in (3.5)-(3.7). Note that the $\hat{T}$ functions have an imaginary part only for $\mathrm{v}\!\cdot\!\mathrm{q}\le(m_b^2+q^2-(m_j+\lambda)^2)/(2m_b)$; this λ dependence of the integration limits cancels all gluon mass dependence upon integration over $\mathrm{v}\!\cdot\!\mathrm{q}$, in the limit $\lambda\to 0$.

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